

## A Chinese Mathematical Classic of the Third Century: *The Sea Island Mathematical Manual* of Liu Hui

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The *Haidao suanjing* [*Sea island mathematical manual*], written by the Chinese mathematician Liu Hui in 263 A.D., consists of nine surveying problems whose solution schemes involve the use of right triangle theory and result in a variety of techniques and formulas for determining distances to inaccessible points. Liu's results were obtained through the use of a prototrigonometry based on the concept of *chong cha*. This paper presents a translation of the *Haidao*'s mathematical exercises and solution formulas and considers some of the implications of this early mathematical work. © 1986 Academic Press, Inc.

< 海島算經 > 是劉徽於公元二六三年寫成的  
有關數學方面的著作，書裡共提出九個測量問題  
，都是利用直角三角形的理論來加以解答的。從  
這些理論可得到幾種測量可望而不可即的目標的  
技巧和方法。這些都是劉徽根據“重差術”的概念  
所得的成果。本文是將< 海島算經 > 的九個問  
題譯成英文，並驗證問題的解法及討論這部算經  
中的一些含意。 © 1986 Academic Press, Inc.

Хайдао суан-синь ("Математика на острове в океане") был написан  
китайским математиком Лю Хуи в 263 году до н.э. Учебник  
содержит девять задач по съемке местности, решение которых  
основано на использовании теории прямоугольного треугольника.

При решении этих задач предлагаются различные приемы и формулы определения расстояния до труднодоступных точек. Лю прибегает к зачаткам тригонометрии, используя понятие "чонг-ча".

В настоящей статье дается перевод задач и формул решения из Хайдао, а также оценивается значение этой ранней математической работы. © 1986 Academic Press, Inc.

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## 1. INTRODUCTION

Much has been said and written about the importance of the *Jiu zhang suanshu* [a]<sup>1</sup> [*Nine chapters on the mathematical art*], a work which marks the beginning of the mathematical tradition in China [1]. Divided into nine sections or chapters, the text comprises 246 problems and their solution schemes, all designed to convey an understanding and application of mathematical ideas in the administration of the state and the functioning of social affairs. This mathematical reference was not the work of an individual or of a particular period, but represented the contributions of many mathematical minds prior to and during the time of the Han Dynasty (221 B.C.–A.D. 220). For generations the *Jiu zhang* remained a standard text for mathematical instruction in China and much of East Asia. Chinese mathematical works written before the 16th century followed the text closely in both style and presentation of mathematical problems. Mathematicians of succeeding generations studied this text carefully, provided commentaries on its contents and methods, and frequently developed their own research around its problems. Among the most eminent followers of this mathematical tradition was the scholar Liu Hui [h].

Liu published a commentary on *Jiu zhang suanshu* in A.D. 263 [*Shi shu*, chap. 16, p. 156]. Nothing is known about the life of Liu Hui except that he was from the State of Wei [k] and flourished during the third century A.D., the Three Kingdom period of Chinese history. In his commentary, Liu gave the problems theoretical verifications, expanding and enriching the text with his own contributions [Ho 1973]. In the preface to the annotated text, Liu mentioned the inadequacy of the ninth chapter, which treats right triangles. This deficiency spurred him to write a new chapter on this subject as an addendum to the original text [Qian 1963, 92]. He called the chapter "double differences" [*chong cha*] [m], explaining that the method of "double differences" was not something new; it had already been used in *Zhou bi suanjing* [n] [*The arithmetic classic of the gnomon and the circular paths of heaven*] to measure the distance to the sun. At the end of the first century Zhang Heng [o] in the *Ling Xien* [p] [*Spiritual constitution of the universe*]

<sup>1</sup> Letters included in brackets refer to characters given in the Chinese Glossary at the end of the article.

mentions the use of the double right-angled triangles, the situation that gives rise to the *chong cha* method [Needham 1959, 104]. Thus Liu Hui was elaborating on a mathematical technique known and accepted in the China of his time. In the preface, he explained that if the horizontal distance to an object were known, the *chong cha* method could be used to find the inaccessible height or depth of that object [Qian 1963, 92]. Then, to illustrate the principle of "double differences," he cited a problem, obtained by generalizing the quantitative account of sun measurements given in the *Zhou bi suanjing*.

Erect two gnomons at the city of Loyang. Let the height [of each gnomon] be eight *chi* [q]. [Both the gnomons erected] in the north-south direction are on the same level. Measure the shadows [of the gnomons] at noon on the same day. The difference in length of the shadows is taken as the *fa* [r]. Multiply the difference in distance of the gnomons by their height and take the result as the *shi* [s]. Divide the *shi* by the *fa* and add to it the height of the gnomon. The result is the height of the sun from the earth. Take the shadow length of the southern gnomon and multiply it by the distance between the gnomons to give the *shi*. Divide the *shi* by the *fa*. The result is the distance of the southern gnomon from the subsolar point in the south at noon. [Qian 1963, 92]

*Chi* is a unit of measure equivalent to a "foot." *Fa* and *shi* are technical terms, indicating divisor and dividend, respectively. Since computation at this time was performed on a counting board with a set of computing rods, the solution instructions are rather mechanical and supply no mathematical justification as to their correctness. Using modern techniques, the problem is easily analyzed. In Fig. 1, the two gnomons are represented by  $\overline{AS}$  and  $\overline{CN}$ , having equal lengths  $h$ . Let the distance  $SN$  between the gnomons be denoted by  $X$ , the length  $SB$  of the shadow of the southern gnomon be  $a_1$ , and the length  $ND$  of the shadow of the northern gnomon be  $a_2$ . At point  $C$  construct  $\overline{CE} \parallel \overline{AB}$  and  $\overline{CR} \parallel \overline{QD}$ . If the height of the sun is represented by  $Y$ , and the distance of the southern gnomon from the subsolar point  $Q$  is  $Z$ , then by using the pairs of similar triangles,  $PRA$  and  $CNE$  and  $PAC$

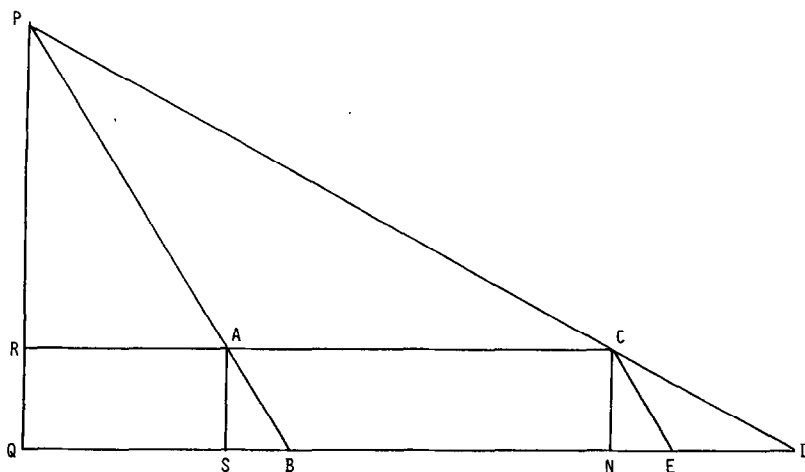


FIGURE 1

and  $CED$ , we obtain  $RP/NC = RA/NE = PA/CE = AC/ED$  or  $(Y - h)/h = Z/a_1 = X/(a_2 - a_1)$ , from which are found  $Y = hX/(a_2 - a_1) + h$  and  $Z = a_1X/(a_2 - a_1)$ , where  $X$  is the difference in distance between the gnomons and  $a_2 - a_1$  is the difference between the lengths of the gnomons' shadows. Hence, for finding the height and distance of an object from a point of observation, an expression of the "double differences" can be employed [2].

In his effort to explain the application of this method of triangulation and to improve upon the right triangle theory given in *Zhou bi suanjing*, Liu Hui devised nine problems and provided their solutions with the aid of diagrams [3]. These problems ask for the height of an island as viewed from the mainland, the height of a tree on a distant hill, the size of a distant walled city, the depth of a ravine, the height of a tower as viewed from a hill, the width of a river mouth as observed from a distance, the depth of a pool of water, the width of a river as seen from a hill, and the size of a city observed from a distant mountain. Realizing that the problems he posed were complex, Liu substantiated his methods by means of a commentary and diagrams. However, when Li Chunfeng [w] (A.D. 602–670) wrote a commentary on Liu's problems, he mentioned neither diagrams nor an existing commentary. Thus, the original diagrams and commentary were lost at an early date, probably before the beginning of the Tang Dynasty (A.D. 618–906). Eventually, the "double differences" chapter was removed from the *Jiu zhang suanshu* and became a separate mathematical work, the *Haidao suanjing* [u] [*Sea island mathematical manual*]. The text's name was derived from its first problem, which involves the measurement of a sea island.

## 2. PRESERVATION AND EVOLUTION OF THE TEXT

In 656 when the Tang government instituted a department of mathematics at its Royal Academy, both the *Jiu zhang suanshu* and *Haidao suanjing* were made part of the *Suanjing shi shu* [v] [*Ten mathematical manuals*], the standard references used by students preparing for official examinations. The required texts were annotated by Li Chunfeng, who also provided commentaries for the candidates. Under official regulations, most of the other texts were prescribed for a year of study, but *Jiu zhang suanshu* and *Haidao suanjing* were allotted three years [*Tang liu dian* [x], chap. 21, p. 106b]. The additional time indicates their significance in the Tang program of mathematical studies. When formal schools were founded in Japan in the eighth century, *Jiu zhang suanshu* and *Haidao suanjing* became prescribed texts in their mathematics curriculum.

During the Song Dynasty (A.D. 960–1278), two versions of the *Haidao suanjing* were published, in 1084 and 1213, but they were lost by the beginning of the Qing Dynasty (A.D. 1644–1911). In their own works, eminent mathematicians of the period, such as Qin Jiushao [bl] and Yang Hui [ap], duplicated and commented upon specific problems of the *Haidao*, but the complete collection of problems, as an independent mathematical entity, became rare. Chu Shijie [bm] included the *Haidao* problems in his *Si yuan yu jian* [bn] [*Precious mirror of the four elements*]

(1303). The collection of problems was also preserved, though in a somewhat haphazard arrangement, in the *Yonglo da dian* [z] [*The encyclopedia of the Yonglo reign-period*] compiled under the Ming rules during the years 1403–1407. Years later, the scholar Dai Zhen [aa] (A.D. 1724–1777) gathered the *Haidao*'s problems from the *Yonglo da dian* and restored the text to its present known form. This reconstituted text was reproduced in the *Wo ying dian* [ab] Palace edition (ca. 1794), then the *Wei po xie* [ac] edition of *Suanjing shi shu* by Kong Jihan [ad], and finally as an appendix to the edition of *Jiu zhang suanshu* by Qu Zhenfa [ae].

During the Qing Dynasty (A.D. 1644–1911), Li Huang [af] (d. 1811) wrote *Haidao suanjing xi chao tu shuo* [ag] [*Detailed diagrammatic explanation of the Sea island mathematical manual*] (published in 1819). Subsequently, Shen Quinpei [ah] produced a work entitled *Chong cha tu shu* [ai] [*Diagrammatic explanation of the "double differences"*]. Both texts used properties of similar triangles to verify the techniques employed in the original version. However, the diagrams provided by Li and Shen appear to contain many extraneous lines, and the texts use techniques that could not have been known to Liu Hui [Liu 1942]. In 1879 another mathematician, Li Liu [am], wrote *Haidao suanjing wei bi* [an] [*Notes on the Sea island mathematical manual*], in which he used the *tian yuan* [ao] ["celestial element"] method to solve the sea island problems. As the *tian yuan* method was a product of the Song-Yuan period (A.D. 960–1368), its use by Liu Hui is indeed doubtful.

In the preface to the *Jiu zhang suanshu* Liu Hui asserted that his mathematical methodology was "to use words to explain the principle and diagrams to illustrate the working" [Qian 1963, 92]. He explains the need to devise the *chong cha* method, adding that he had provided a commentary for it. The original version of *Haidao suanjing* appeared to be a comprehensive text with diagrams, derivations, and the author's commentary. But by the 13th century all that remained were the problems accompanied by their solutions. Even a great mathematician like Yang Hui was perplexed by "the setting of the methods and problems in the *Haidao*" [Lam 1977, 14]. When he intended to include the *Haidao* problems in his *Xu gu zhai qi suanfa* [az] [*Continuation of ancient mathematical methods for elucidating the strange (properties of number)*] of 1275, Yang had "to place a small diagram of the sea island before him so that he was able to understand a little of the method employed by his predecessors" [Lam 1977, 179]. However, it is difficult to say whether Yang Hui was referring to a copy of Liu Hui's original diagram or one that he had drawn himself. In his deliberations of these problems, he provided a theoretical proof of the *chong cha* method.

Since 1926, when Li Yan published his research on the origins and applications of the *chong cha* method, the text of *Haidao suanjing* has attracted considerable attention among Chinese historians of mathematics [4]. Li Yan's exposition of Liu Hui's formulas was based on the properties of similar triangles, and this approach has remained for almost half a century the primary basis for Chinese research on the methods of the *Haidao* [Xu 1954, 15–23]. Bai Shangshu [bo], in his recent

study of the *Haidao* problems, surveyed historical explanations of the *chong cha* method as provided by Chinese mathematicians from the Song-Yuan period (A.D. 960–1368) onward. On the basis of this survey, Bai is convinced that Liu Hui used a theory of proportions involving similar right triangles to derive proofs for the *Haidao* formulas [Bai 1982]. Wu Wenjun [f], however, opposes this theory and offers a different geometric insight to interpret Liu Hui's work [Wu 1982a,b]. Convinced that ancient Chinese mathematicians did not employ concepts of angles or parallel lines in their theories, Wu bases his understanding of Liu's methods on an analysis of Zhao Shuang's [ay] "diagram of solar height" given in the *Zhou bi suanjing*. He speculates that Liu, in studying this diagram, had perceived that the complements of the angles formed by the diagonals of a given rectangle are pairwise congruent, citing Liu Hui's commentary to Chapters 5 and 9 of the *Jiu zhang suanshu* as evidence for this hypothesis. Further, Wu asserts that the "small diagram of the sea island," which Yang Hui referred to in his *Xu gu zhai qi suanfa*, may well have been a copy of Liu Hui's original diagram. Thus, Wu believes that his reconstructed geometric proofs for the *Haidao suanjing* formulas are similar to those suggested by Yang Hui.

Research in the West on the *Haidao suanjing* has been singular and sparse. Mikami [1912] published an English-language translation of the first three problems. But the first complete translation of the problems into an Occidental language (French) was made by van Hée [1932], who had already published a translation of the first three problems of *Haidao* [van Hée 1920]. Van Hée's discussion of the problems was brief and contained little about the mathematical methods involved. More recently, Ho [1973] has provided translations of the first, fourth, and seventh problems. E. I. Berezkina has published (in Russian) [1974] a complete translation of the problems and a verification of their solution formulas. In a still more recent work, she provides a detailed discussion of problems 1, 3, 4, and 8 [Berezkina 1980].

A complete English-language translation of Liu's *Haidao* problems and an analysis of its contents follow.

### 3. THE PROBLEMS OF THE SEA ISLAND MATHEMATICAL MANUAL

This translation of the text is based on the contents of Qian Baocong's edition of *Suanjing shi shu*. Li Chunfeng's commentary is omitted because it sheds little light on Liu Hui's solution methods. The format of all problems is similar: a paragraph with the statement of the problem, followed by the answer and, finally, a paragraph describing the solution procedure. To indicate the sequential ordering of presentation and for purposes of referencing, numbers have been assigned to the problems. The metrology employed in the series of problems is

$$li [ba] = 1800 \text{ chi}$$

$$chang [bb] = 10 \text{ chi}$$

$$bu [bc] = 6\ chi$$

$$chi = 10\ cun [bd].$$

(1) Now for [the purpose of] looking at a sea island, erect two poles of the same height, 3 *chang* [on the ground], the distance between the front and rear [poles] being a thousand *bu*. Assume that the rear and front poles are aligned [with the island]. By moving away 123 *bu* from the front pole and observing the peak of the island from ground level, it is noticed that the tip of the front pole coincides with the peak. Then by moving backward 127 *bu* from the rear pole and observing the peak of the island from ground level again, the tip of the back pole also coincides with the peak. What is the height of the island and how far is it from the pole?

The height of the island is 4 *li* 55 *bu*, it is 102 *li* 150 *bu* from the pole.

Multiply the distance between poles by the height of the pole, giving the *shi*. Taking the difference in distance from the points of observation as the *fa* to divide [the *shi*], and adding what is thus obtained to the height of the pole, the result is the height of the island.

To find the distance of the island from the front pole, multiply [the distance of the] backward movement from the front pole by the distance between the poles, giving the *shi*. Taking the difference in distance at the points of observation as the *fa* to divide [the *shi*], the result is the distance of the island from the pole in *li*.

(2) Now for [the purpose of] looking at a pine tree of unknown height growing on a hill, erect two poles of the same height 2 *chang* [on the ground], the distance between the front and the rear [pole] being 50 *bu*. Assume that the rear pole and front pole are aligned [with the pine tree]. By moving backwards 7 *bu* 4 *chi* from the front pole and observing the top of the pine tree from ground level, it is observed that the tip of the pole coincides with the top of the pine tree. Sighting again at the foot of the pine tree, the line of observation intersects the pole at a point 2 *chi* 8 *cun* from the tip of the pole. Once more by moving backwards 8 *bu* 5 *chi* from the rear pole and observing the top of the pine tree from ground level, it is seen that the tip of the pole also coincides with the top of the pine tree. What is the height of the pine tree and how far is the hill from the pole?

The height of the pine is 12 *chang* 2 *chi* 8 *cun*. The hill is 1 *li* 28½ *bu* from the pole.

Multiply the distance between poles by the observation measurement from the top of the pole, giving the *shi*. Taking the difference in distance at the points of observation as the *fa* to divide [the *shi*] and adding to it the observation measurement along the pole, the result is the height of the pine tree.

To find the distance of the hill from the pole, place the distance between poles [on the counting-board] and multiply it by the distance of the rear movement from the front pole, giving the *shi*. Taking the difference in distance from the points of observation as the *fa* to divide [the *shi*], the result is the distance of the hill from the pole.

(3) Now for [the purpose of] looking southward at a square [walled] city of unknown size, erect two poles 6 *chang* apart in the east–west direction such that they are standing at eye level and are joined by a string. Assume that the eastern pole is aligned with the southeastern and northeastern corners of the city. By moving northward 5 *bu* from the eastern pole and sighting on the northwestern corner of the city, the line of observation intersects the string at a point 2 *chang* 2 *chi* 6½ *cun* from its eastern end. Once again moving northward 13 *bu* 2 *chi* from the pole and sighting on the northwestern corner of the city, the corner coincides with the western pole. What is the length of the squared city and how far is the city from the pole?

The length of the squared city measures 3 *li* 43¾ *bu*. The city lies 4 *li* 45 *bu* from the pole.

Multiply the final rear movement from the pole by the observation measurement obtained

on the string and divide [the product] by the distance between poles. What is thus obtained is the shadow difference. Subtracting the initial rear movement from the pole [from the shadow difference], the remainder is the *fa*. Place the final rear movement from the pole [on the counting-board] and subtract the initial rear movement from the pole. The remainder is multiplied by the observation measurement along the string, giving the *shi*. Dividing the *shi* by the *fa* yields the length of the squared city.

To find the distance [of the city] from the pole, place the final rear movement from the pole [on the counting-board] and subtract the shadow difference. The remainder is multiplied by the initial rear movement from the pole, giving the *shi*. Dividing the *shi* by the *fa* yields the distance of the city from the pole.

(4) Now for [the purpose of] looking into a deep valley, set up a carpenter's square of height 6 *chi* on the tip of the valley. Sighting on the bottom of the valley from the tip of the carpenter's square, the line of observation intersects the base at a point 9 *chi* 1 *cun* from the corner of the square. Set up another similar carpenter's square above [the first]; the distance between the bases of the [two] squares is 3 *chang*. Sighting on the bottom of the valley from the tip of the upper square, the line of observation intersects the base at a point 8 *chi* 5 *cun* [from the corner of the upper square]. How deep is the valley?

41 *chang* 9 *chi*.

Place the distance between the carpenter's squares [on the counting-board] and multiply it by the [observation measurement obtained along] the base of the upper square, giving the *shi*. Subtract this [observation measurement obtained along the] base from the upper square from [that obtained along] the base of the lower square, and take the remainder as the *fa*. Perform the division, and subtract the height of the carpenter's square from it [i.e., the quotient]; the result is the depth of the valley.

(5) Now for [the purpose of] observing a building on the level ground from a mountain, erect a carpenter's square of height 6 *chi* on the mountain. Sighting obliquely downwards on the foot of the building from the tip of the square, the line of observation intersects the base at a point 1 *chang* 2 *chi* [from the corner of the square]. Set up again another similar carpenter's square above [the first]; the distance between the bases of the [two] squares is 3 *chang*. Sighting obliquely downwards on the foot of the building from the tip of the upper square, the line of observation intersects the base at a point 1 *chang* 1 *chi* 4 *cun* [from the corner of the square]. Erect another small pole at the point of intersection and sight obliquely downwards on the pointed gable of the building from the tip of the square again; the line of observation intersects the pole at a point 8 *cun* from its foot. What is the height of the building?

8 *chang*.

Subtract [the observation measurement obtained along] the base of the upper square from [that obtained along] the base of the lower square, and take the remainder as the *fa*. Place the distance between the carpenter's squares [on the counting-board], multiply it by [the observation measurement obtained along] the base of the lower square and divide the product by the height of the square. The result obtained is multiplied by the observation measurement obtained on the [vertical] pole, giving the *shi*. Dividing the *shi* by the *fa* yields the height of the building.

(6) Now for [the purpose of] looking towards the southeast at the mouth of a river, erect two poles in the north-south direction; the distance between them is 9 *chang* and they are joined along the ground by a string. Facing the west and moving away 6 *chang* from the north pole to observe the southern bank of the mouth of the river at ground level, the line of observation intersects the string at a point 4 *chang* 2 *cun* from the north end of the pole. Sighting on the northern bank [from the same position], the line of observation intersects the string at a point 1 *chang* 2 *chi* [from the previous intersection]. Moving [westward] again by 13 *chang* 5 *chi* from the [north] pole and observing the southern bank of the river's mouth, it



is seen that this view coincides with the pole in the south. What is the width of the river's mouth?

1 li 200 bu.

Multiply the [first] observation measurement obtained along the string by the final rear movement from the pole and divide [the product obtained] by the distance between poles. Subtract from this the initial rear movement from the pole, giving the remainder as the *fa*. Then multiplying the difference between the initial and final rear movements from the pole by the [second] observation measurement obtained along the string, the result is the *shi*. Dividing the *shi* by the *fa* yields the width of the river's mouth.

(7) Now for [the purpose of] looking into a deep abyss containing a clear pool of water with white stones at the bottom, hold a carpenter's square on the edge of the abyss. Assume that the height of the square is 3 *chi*. Sighting obliquely downwards on the water level, the line of observation intersects the base at a point 4 *chi* 5 *cun* [from the corner of the square] while the line of observation for the white stone intersects the same base at 2 *chi* 4 *cun*. Erect another similar carpenter's square above [the first] such that it is 4 *chi* from the lower one. Sighting obliquely downwards again on the water level and the white stone, the line of observation intersects the base at 4 *chi* and 2 *chi* 2 *cun* [from the corner of the upper square], respectively. What is the depth of the water?

1 *chang* 2 *chi*.

Set up the observation measurements of the water level obtained along the bases of the upper and lower carpenter's squares and subtract one from the other. The remainder is then multiplied by the observation measurement of the stone obtained along the base of the upper square. The result is taken as the upper rate. Obtain the difference of the observation measurements of the stone taken along the bases of the upper and lower squares and multiply the remainder by the observation measurement of the water taken along the base of the upper square. The result is taken as the lower rate. Subtracting one rate from the other and multiplying the remainder by the distance between the squares gives *shi*. Multiply the two differences [obtained previously] together and take [this product] as the *fa*. Dividing the *shi* by the *fa* yields the depth of the water.

(8) Now [for the purpose of] observing a river in the south from a mountain, erect a carpenter's square on the mountain. Assume that the height of the square is 1 *chang* 2 *chi*. Sighting obliquely downwards on the southern bank [of the river] from the tip of the square, the line of observation intersects the base at a point 2 *chang* 3 *chi* from the corner of the square. Sighting again on the northern bank [from the same position], the line of observation intersects the same base at 1 *chang* 8 *cun* [from the point of previous intersection]. Facing the north and moving away 22 *bu* [from the previous position of observation], climb up a cliff which is 51 *bu* higher and erect another similar carpenter's square there. Sighting obliquely downwards on the southern bank from the tip of this square, the line of observation intersects the base at a point 2 *chang* 2 *chi* from the corner of the square. What is the width of the river?

2 li 102 bu.

Multiply the [observation measurement of the southern bank obtained along the] base of the lower carpenter's square by the height of the square and divide the [observation measurement of the same bank obtained along the] base of the upper square. Subtract the height of the square from the result obtained, thus giving the *fa*. Multiply the northward movement by the height of the square and divide it by the [observation measurement of the southern bank obtained along the] base of upper square. Subtract the result obtained from the upward movement, and multiply the remainder by the observation measurement of the northern bank obtained along the base of the lower square, thus giving the *shi*. Dividing the *shi* by the *fa* yields the width of the river.

(9) Now for [the purpose of] observing a city in the south from a mountain, set up a

carpenter's square on the mountain. Assume that the height of the square is 3 *chi* 5 *cun*, and that the tip of the square is aligned with the southeastern corner and with the northeastern corner of the city. Sighting on the northeastern corner [of the city from the tip of the square], the line of observation intersects the base at a point 1 *chang* 2 *chi* [from the corner of the square]. Set up [separately] a horizontal arm [of another carpenter's square] at the point of intersection [such that the arm is perpendicular to the base]. Sighting on the northwestern corner [of the city from the tip of the square], the line of observation intersects the horizontal arm at a point 5 *chi* [from the point of intersection]. Sighting on the southeastern corner [from the tip of the square], the line of observation intersects the base at a point 1 *chang* 8 *chi* [from the corner of the square]. Set up again another similar carpenter's square above, assuming that the two squares are 4 *chang* apart. Sighting on the southeastern corner [of the city from the tip of the square], the line of observation intersects the base at a point 1 *chang* 7 *chi* 5 *cun* [from the corner of this upper square]. What is the width and length of the city?

The length [of the city] in the north-south direction is 1 *li* 100 *bu*; the width in the east-west direction is 1 *li* 33½ *bu*.

Multiply the observation measurement of the southeastern corner obtained along the base of the lower square by the height of the square and divide [the produce obtained] by the [observation measurement obtained along the] base of the upper square. Subtract the height of the square from what is obtained, thus giving the *fa*. Subtract the observation measurement of the northeastern corner along the base of the lower square from that of the southeastern corner and multiply the remainder obtained by the distance between the squares, giving the *shi*. Dividing the *shi* by the *fa* yields the length [of the city] in the north-south direction. To find the width of the city, multiply the distance between the squares by the observation measurement obtained along the horizontal arm, giving the *shi*. Dividing the *shi* by the *fa* yields the width of the city in the east-west direction.

#### 4. ANALYSIS AND DISCUSSION OF THE CONTENTS OF *HAIDAO SUANJING*

While intended as a revision and extension of the ninth chapter of the *Jiu zhang*, the nine problems that constitute the *Sea Island Manual* are distinctly more advanced in their mathematical conceptualization of right triangle theory than those of the *Jiu zhang*. Thus it is not surprising that eventually they were separated from the text to stand as an independent work.

The twenty-four problems of the *Jiu zhang* involving right triangles form a primer on right triangle theory [Swetz & Kao 1977]. The "Pythagorean proposition" and its applications are introduced and reinforced by a variety of sometimes fanciful problems. Obtaining solutions to a few of these problems involves the use of simple proportions resulting from a similarity of right triangles. By contrast, the solution of each *Haidao* problem requires two independent sets of similar right triangles; the simultaneous solution of the resulting pairs of proportions is the focus of the *chong cha* technique. Taking at face value the literal translation of the phrase "double difference," along with Liu's prefatory remarks and the solution formula for his first problem, one may infer that problems placed in the *chong cha* category consist of those whose solutions require the securing and manipulation of two numerical differences; however, this is not necessarily the case. Not all the *Haidao*'s problems contain such numerical double differences in their solution

schemes, indicating that the term *chong cha* had a broader meaning. Thus, Mikami interpreted *chong cha* to mean repeated or double applications of proportions [Mikami 1912, 35], while van Hée, who doubted the literal meaning of “double difference,” considered it to be an application of double proportions [van Hée 1932, 267]. Berezkina translated *chong cha* as “double level differences,” implying the use of physical measurements taken at two difference levels or locations on the same line—a situation that would naturally result in the use of double proportions [Berezkina 1980, 278]. Even Chinese commentators questioned the scope of Liu’s technical designations: Yang Hui (A.D. 1275), in an examination of Liu’s work, noted that “men of the past changed the names of their methods from problem to problem” [Lam 1977, 345]. If this is indeed the case then, at most, *chong cha* can be taken to mean a general class of problems concerning right triangles.

The last six problems of the *Jiu zhang* concern land surveying situations, and in the original extension of the ninth chapter of the *Jiu zhang*, Liu used only such problems. Thus all the problems of the *Haidao* involve surveying calculations; four of them indicate the use of sighting poles, *biao* [be], and five require the use of a carpenter’s square, *ju* [bf]. Both instruments are mentioned in Chinese mythology in association with the founding of science and mathematics, and they had a long history of use in China before the time of Liu Hui. By associating mathematics with surveying, Liu was considering an important and relevant aspect of the applied mathematics of his time. Accurate surveying practices and formulas were essential for both state and military affairs [5]. But as a guide to surveying strategies, the *Haidao*’s problems seem more academic than practical: in problem 1, the two observing poles should be erected with their bases at sea level to ensure the accuracy of the height and distance of the island using the given formulas; in problems 1 and 2, the observer must take awkward sightings from ground level rather than eye level; in problem 5, observations are taken on a perfectly vertical wall, an architectural feature uncommon in China of the third century; and in problem 7, no allowances are made for the refractive index of the water, rendering the two observations inaccurate. These examples suggest that Liu used the problems arising in surveying to demonstrate a rather powerful right angle computational theory. We may conclude, therefore, that Liu’s work was intended as a thesis in mathematics, rather than an enhancement of surveying techniques.

It is easy to associate *chong cha* computational procedures with trigonometric thinking. Indeed, Alexander Wylie, the first Western commentator of the *Haidao*, described its contents to be “nine problems in practical trigonometry” [Wylie 1867]; Needham, in a more recent view of the work, referred to *chong cha* as “a kind of empirical substitute for trigonometric functions” [Needham 1959, 109]. The apparent trigonometric aspects of the problems deserve examination. Consider the modern verification of the solution for problem 3:

In Fig. 2, *GDOH* is a square walled city. Sighting poles are placed at points *F*



In this problem, Liu defines the “shadow difference,” *jing cha* [bj], to be  $JC - (AC)(EC)/FC$ . If in Fig. 2, we let  $\angle AFC = \alpha$ , then  $\tan \alpha = AC/FC$  and  $JC$  becomes the length of the projection of  $\overline{EC}$  along  $\overline{AC}$ ; i.e., in a physical sense  $JC$  may be described to be the shadow of  $\overline{EC}$  caused by light rays parallel to  $\overline{GA}$ . This shadow difference ratio supplies a computational link between the two sets of similar triangles associated with this problem, and it facilitates a solution. While a modern student of mathematics might resolve the construction of  $\overline{JE}$  to establish a relationship between  $\overline{EC}$  and  $\overline{FC}$ , Liu devised the ratio known as the “shadow difference” and implicitly employed the tangent function. Similar ratios appear in the solutions to problems 5, 6, 8, and 9. The use of “shadow functions”—the tangent and cotangent—were familiar to many ancient peoples who observed the heavens with the aid of a vertical staff or gnomon [6]. Solutions of problems given in the *gou gu* [bk] section of the *Jiu zhang* also utilized ratios that today would be recognized as tangents of given angles; however, Liu’s use of tangent ratios to solve rather intricate geometrical problems is far in advance of the procedures offered in the *Jiu zhang* and, although limited, his techniques may be thought of as a prototrigonometry.

Any attempt to understand Liu’s mathematical thinking would require an examination of his derivations for the proposed solutions; unfortunately no such derivations exist. The *Haidao suanjing*, as it survives today, contains no information about the methods of derivation, and few Chinese commentators on Liu’s work have attempted to unravel his possible methodology. In *Shu shu Jiu zhang* [bg] [*Mathematical treatise in nine sections*] (A.D. 1247), Qin Jiushao [bl], the Song scholar, presented several problems that emulate Liu’s; however, as Libbrecht has pointed out, Qin possessed little real understanding of Liu’s methodology and made several serious mistakes in his presentation [Libbrecht 1973, 122–149]. A contemporary of Qin, Yang Hui, who was mainly interested in applied mathematics, compiled several works on this subject. In *Xu qu zhai qi suanfa*, Yang gave four problems on the measurement of inaccessible heights. Included in this series was Liu’s first problem of the sea island, upon which Yang commented and provided a justification of one of the solution formulas. Given the sea island situation, a summary of Yang’s derivation follows:

In Fig. 3,  $\triangle CBM \cong \triangle CAB$  and  $\triangle CDN \cong \triangle CJD$ ; therefore area  $(\triangle CBM - \triangle CDN) = \text{area}(\triangle CAB - \triangle CJD)$ . Since  $\triangle BLD \cong \triangle DEB$  and area (trapezoid  $DBMN = \text{trapezoid } JABD$ ), area  $(DBMN - BLD) = \text{area}(JABD - DEB)$ ; therefore area (rectangle  $DLMN$ ) = area (rectangle  $JAED$ ). In rectangle  $CAHO$  area  $(JAGF) = \text{area}(PFKO)$  and in rectangle  $PGBM$  are  $(FGED) = \text{area}(DLMN - PFKO)$  (1); but area  $(FGED) = (GE)(ED)$ , and from (1) area  $(FGED) = (ML) \times (EB - GH)$ . Therefore  $ML = (GE)(ED)/(EB - GH)$ , and since  $ML = CJ$ , where  $CA = CJ + JA$  and  $JA = ED = FG$ , it follows that  $CA = (GE)(FG)/(EB - GH) + FG$ .

Although this derivation was carried out in the 13th century, Yang’s methods reflect traditional geometric–algebraic thinking. Since this form of mathematical

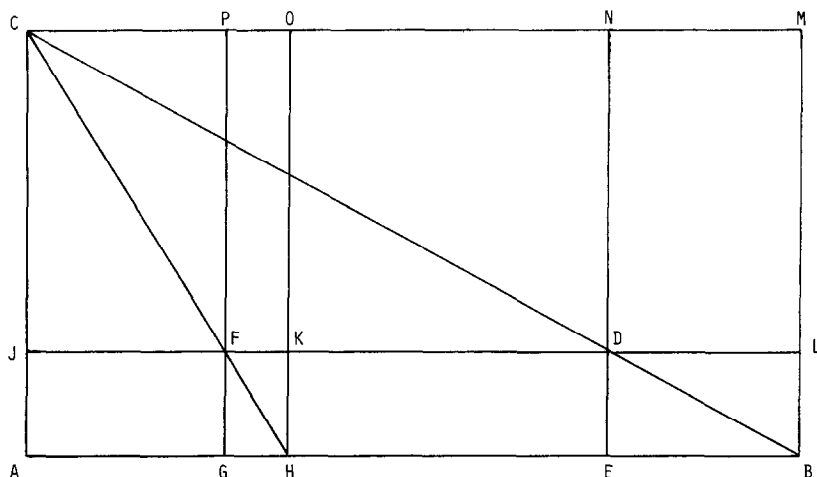


FIGURE 3

thinking had changed little in China in the course of a thousand years, Yang's derivation may well provide some insights into the techniques used by Liu Hui [7].

## 5. CONCLUSION

Problems involving right triangles and their uses appeared in ancient China as early as 1000 B.C. [Gillon 1976; Ang 1977]. The *Haidao suanjing* with its right triangle computations remains a classic of traditional Chinese mathematics. For over a millenium, its influence was felt in the work of Eastern mathematicians. Although the *Haidao suanjing* was a product of third-century China, a comparative retrospect helps to establish its significance in the history of mathematical achievement.

Liu initiated his discussion of measuring distances by considering a problem from the *Zhou bi suanjing* which sought to ascertain the distance of the sun from the earth. Although the dating of the *Zhou bi* is controversial, it is probably safe to conclude that the inclusion of such problems indicates that the Chinese were concerned with finding the solar distance prior to the second century B.C. It is interesting to note that at about this same time, the Greeks made similar attempts to measure the heavens. Aristarchus (ca. 310–230 B.C.) used ratios to determine celestial distances. Like his Chinese contemporaries, he obtained inaccurate results due to false assumptions.

While there is no doubt that the Greeks used mathematics effectively in surveying their structures and cities, no extant records of this practice exist. Greek mathematical methodology was adopted and modified by the Romans, who used it widely in surveying problems arising in military campaigns and the settlement of conquered territories. The Romans compiled surveying manuals which were broad in scope and included many aspects of survey practice: physical; legal; and

alteracuneque A. Quare per seque pnt dca: deo lura eis oppoia  
F A, A B trunt equalia. Quod dictum fuit erat.

PROPOGITO. XI.

Eadem oblationem diametralium spici implere positi & signa quorundam aliorum sicut ad illud planum erecti, ita in alium suo edificij perpendiculariter ad illud planum erecti, ita tamen, ut ad signum plani, & ad basim edificij accedendi possint, per Quod auri Geometricum indagare.

**P**ortis vero per quadratum Geometricum distantiam diametralem A B in hunc modum venari. Posito Quadrato ad perpendicularum ob-  
turbis figuram B per dioptram, et notabis punctum ab ea scilicet, quod  
quidem erit vel in latere D E vobis rectis, vel in linea A B vobis in-  
ducis, vel in latere E F vobis rectis.

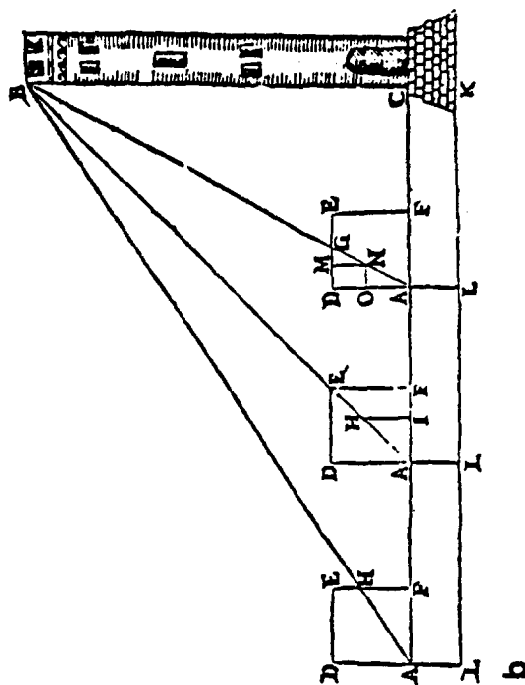
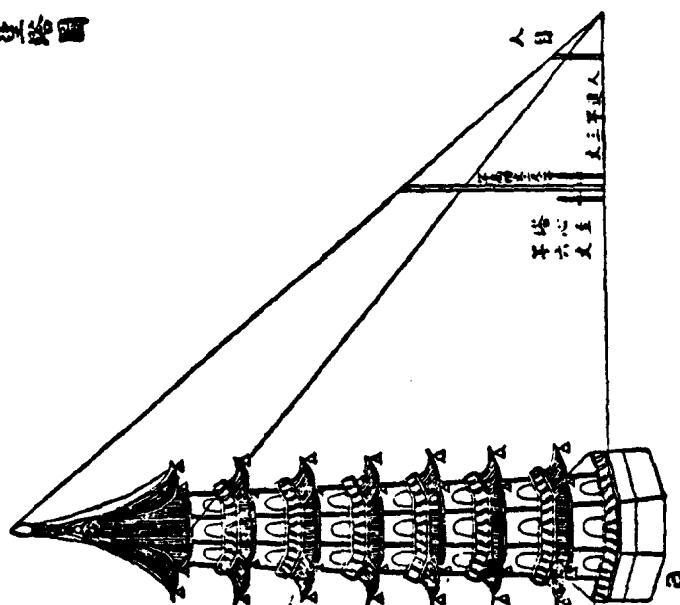


FIG. 4. Tower problems. (a) Reprinted from Qin Juishao, *Shu shu jiu zhang* [bs] (+1247). (b) Reprinted from G. A. Magini, *De plantis triangularis, Liber unicus* (1592).

theoretical, which included an introduction to applied geometry [Dilke 1967, 1971; Blume et al., 1967]. This geometric knowledge became a primary source for medieval scholarship on the subject. Gerbert's (940–1003) *Gerberti isagoge geometriae* became the first European, medieval work to include practical geometric applications, among them problems whose descriptions and solutions were similar to those discussed in the *Haidao*. This work was followed by Savasorda's (ca. 1070–1130) *Liber embadorum*, a text compiled from Spanish sources, which contained primarily geometric applications, such as finding lengths, heights, and areas; there were also the practical geometries, such as that of Hugonis [Baron 1956; Victor 1979]. However, it was Leonardo of Pisa's *Practica geometriae* (1220) that placed surveying by trigonometric methods on a firm mathematical basis [Zeller 1946] [8]. Leonardo's work was strongly influenced by both Gerbert's *Geometriae* and Plato of Tivoli's 1145 translation of the *Liber embadorum*. With a fuller appreciation of an angle as a mathematical entity and the adoption and use of such instruments as the quadrant and astrolabe, European surveying became a trigonometric activity and an extensive literature on the subject appeared [Kiely 1947]. Illustrations from Renaissance mathematical texts bear striking similarities to the problem situations that preoccupied Chinese mathematicians [Vogel 1983]. See Fig. 4.

From this examination of the contents and methods of the *Haidao suanjing* and a review of early Western sources on applied geometry and land surveying, it appears that Chinese efforts in applying mathematical principles to surveying situations surpassed those achieved in the West until the time of the European Renaissance [Victor 1979].

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### NOTES

1. See, for example, [Qian 1964, 28–56; Li & Du 1963, 45–75; Wu 1982a; Needham 1959 3, 24–27; Wang 1956]. The text has also been translated into German [Vogel 1968] and Russian [Berezkina 1957].

2. The “double difference” method of determining the solar height and distance from a point of observation on the ground would be accurate provided that the earth were flat. Traditional Chinese cosmology held to the theory of a flat earth. Although some early schools of cosmology did propose a spherical earth, it was not until the second half of the 17th century that this theory was formalized. It should be noted that the method of “double differences” is correct if sightings are taken on an object relatively close by. Liu Hui cited this example merely to illustrate the principle of “double differences.”

3. *Sui shu*, Chap. 24, p. 11a, states that Kiu Hui composed a chapter of *Jiu zang chong cha tu* [1] [Diagrams of “double differences” in the nine chapters].

4. Li Yan's article on “double differences” first appeared in *Xue yi* [as] [Li 1926] and subsequently was republished in several major collections of material on the history of Chinese mathematics [Li 1933, 1954].

5. Pei Xiu [bh] (A.D. 224–271), the “father of Chinese cartography,” flourished during the time of Liu Hui, as did general Deng Ai [bi] of Liu's own kingdom of Wei who always “estimated the height



and distances by finger-breadths before drawing a plan of the place and fixing the position of his camp'' [Needham 1959, 572].

6. For example, problems 56–60 of the Rhind papyrus (1650 B.C.) reveal that the Egyptians of that period used a mathematical ratio called *seqt* to designate inclination [Gillings 1972, 185; Chace et al. 1927]. Today, the *seqt* would be recognized as the cotangent of an angle. Written records of the Zhou Dynasty (1122–255 B.C.) indicate the existence and use of a gnomon shadow template to calibrate horizontal shadows cast by a vertical staff; thus the Chinese implicitly employed the tangent function in astronomical calculations for hundreds of years before Liu's time. The earliest known systematic compilation of tangent functions can be traced to China [Cullen 1982]. Also see the discussion in [Karpinski 1928; Bond 1921–1922]. Evidence of Hindu considerations with shadow reckoning can be found in Colebrooke's translation of Brahmagupta's works [Colebrooke 1817, Chap. 11; Amma 1979, Chap. 10]. Islamic use of shadow functions is considered in [Kennedy 1976, Chap. 28].

7. The contents and techniques of the *Jiu zhang* dominated the mathematics scene in China for over a thousand years. Mathematics was devoted to simple problem solving and if the technique worked there was little reason to change it.

8. Leonardo's *Practica* consists of eight chapters, two of which, Chapters 3 and 7, are specifically devoted to the needs of surveyors. In Chapter 3, Leonardo describes instrumental methods and computational techniques involving the use of a Ptolemaic chord table. The seventh chapter discusses the use of a quadrant in angle measurement and the determination of inaccessible distances and develops rules of surveying based on the similarity of triangles.

## REFERENCES

- Amma, T. A. S. 1979. *Geometry in ancient and medieval India*. Delhi: Motilal Banarsidass.
- Ang, T. S. 1977. Chinese interest in right-angled triangles. *Historia Mathematica* **5**, 253–266.
- Bai, S. S. 1982. *Liu Hui Suanjing Zao shude Tanta* [bp]. *Kojishi Wenji* [bq] **8**, 79–87.
- Baron, R. 1956. Hugonis de Sancto Victore "Practica geometriae." *Osiris* **12**, 176–224.
- Berezkina, E. I. 1957. Drevnekitaisku traktat matematika v devyati knigakh. *Istoriko-Matematicheskie Issledovaniya* **10**, 423–584.
- 1974. Dva teksta Lyu Khueya po geometrii. *Istoriko-Matematicheskie Issledovaniya* **19**, 231–248.
- 1980. *Matematika Drevnogo Kitaya*. Moscow: Nauka.
- Blume, F., et al. 1967. *Die Schriften der römischen Feldmesser*. 2 vols. Hildesheim: Georg Olms Verlagsbuchhandlung.
- Bond, J. D. 1921. The development of trigonometric methods down to the close of the XVth century. *Isis* **4**, 295–323.
- Chace, A. B., et al. (eds.). 1927–1929. *The Rhind mathematical papyrus*. 2 vols. Oberlin.
- Colebrooke, H. T. 1817. *Algebra with arithmetic and mensuration from the Sanscrit of Brahmagupta and Bhascara*. London: Murray.
- Cullen, C. 1982. An eighth century Chinese table of tangents. *Chinese Science* **5**, 1–33.
- Dilke, O. A. W. 1967. Illustrations from Roman surveyors' manuals. *Imago mundi* **21**, 9–29.
- 1971. *The Roman land surveyors*. New York: Barnes & Noble.
- Gillings, R. J. 1972. *Mathematics in the time of the pharaohs*. Cambridge, Mass.: MIT Press.
- Gillon, B. S. 1976. Introduction, translation, and discussion of Chao Chun-Ch'ing's "Notes to the diagrams of short legs and long legs of squares and circles." *Historia Mathematica* **4**, 253–293.
- Ho, P. Y. 1973. Liu Hui. In *Dictionary of scientific biography*, Vol. 8, pp. 418–425. New York: Scribner's.
- Karpinski, L. 1928. The unity of Hindu contributions to mathematical science. *Scientia* **43**, 381–388.

- Kennedy, E. S. 1976. *The exhaustive treatise of shadows by Abu al-Rayhān Muhammad b. Ahmad al-Bīrūnī*. 2 vols. Aleppo: Institute for the History of Arabic Science.
- Kiely, E. R. 1947. *Surveying instruments: Their history and classroom use*. New York: Teachers College Press.
- Lam, L. Y. 1977. *A critical study of the Yang Hui Suan Fa* [aq]. Singapore: University Press.
- Li, Y. 1926. Chong cha shu yuanliu ji qi xin zhu [ar] [The origins of the method of double differences with a new commentary]. *Xue yi* [as] 7, 1–15.
- 1933. *Zhong suan shu lun cong* [at] [*Collected essays on Chinese mathematics*], Vol. 1. Shanghai: Commercial Press.
- 1954. *Zhongguo gudai shuxue shiliao* [au] [*Materials for the historical study of ancient Chinese mathematics*]. pp. 126–134. Shanghai.
- Li, Y., & Du, S. [d]. 1963. *Zhongguo gudai shuxue jian shi* [e] [*A brief history of Chinese mathematics*]. Peking.
- Libbrecht, U. 1973. *The Shu-shu chiu-chang of Ch'in Chiu-shao*. Cambridge, Mass.: MIT Press.
- Liu, C. N. [aj]. 1942. Haidao suanjing yuanliu kao [ak] [*An investigation into the origins of the Sea-island mathematical manual*]. *Yi shi bao wen shi fu kan* [al], No. 21.
- Mikami, Y. 1912. *The development of mathematics in China and Japan*. Abhandlungen zur Geschichte der mathematischen Wissenschaften, No. 30. Leipzig.
- Needham, J. 1959. *Science and civilization in China*, Vol. 3. Cambridge: University Press.
- Qian, B. [b] (ed.). 1963. *Suanjing shi shu* [l] [*The ten mathematics classics*]. Peking.
- 1964. *Zhongguo shuxue shi* [c] [*History of Chinese mathematics*]. Peking.
- Shi shu* [i]. [Official history of the Sui Dynasty]. *Si bu cong kan* [j], ed.
- Swetz, F. J., & Kao, T. I. 1977. *Was Pythagoras Chinese? An examination of right triangle theory in ancient China*. London: Pennsylvania State Univ. Press.
- Tang liu dian* [x]. [*The institutes of the Tang Dynasty*], Qin ding si ku quan shu [y], ed.
- van Hée, L. 1920. Le Hai-Tao Souan-King de Lieou. *T'oung pao* 20, 51–60.
- 1932. Le classique de l'île maritime, ouvrage chinois du III<sup>e</sup> siècle. *Quellen und Studien zur Geschichte der Mathematik* 2, 255–258.
- Victor, S. K. 1979. *Practical geometry in the High Middle Ages*. Philadelphia: Amer. Philos. Soc.
- Vogel, K. 1968. *Chiu Chang Suan Shu: Neun Bücher arithmetischer Technik*. Braunschweig: Vieweg.
- 1983. Ein Vermessungsproblem reist von China nach Paris. *Historia Mathematica* 10, 360–367.
- Wang, L. 1956. *The Chiu Chang Suan Shu and the history of Chinese mathematics*. Ph.D. dissertation, Cambridge University.
- Wu, W. [f] (ed.). 1982a. *Jiu zhang suanshu yu Liu Hui* [g] [*Nine chapters on the mathematical art and Liu Hui*]. Peking.
- 1982b. Haidao suanjing gu zheng kao yuan [ax] [*Investigations into the original proofs of the Sea-island mathematical manual*]. *Jiu zhang*, 162–181.
- 1982c. Wo guo gudai cewang zhi xue chongcha lilun pingji jian ping shuxueshi yanjiu zhong mo xie fangfa wenti [br] [*On the theory of chang cha in ancient survey and the methodology of the research on the history of Chinese mathematics*]. *Kojishi wenji* [bq] 8, 10–30.
- Wylie, A. 1867. *Notes on Chinese literature*. Shanghai.
- Xu, C. [av] 1954. *Zhong suan jiade jiho xue yanjiu* [aw] [*The study of geometry by Chinese mathematicians*]. Peking.
- Zeller, M. C. 1946. *The development of trigonometry from Regiomontanus to Pitiscus*. Joliet, Ill.: Sisters of St. Francis of Mary Immaculate.

## GLOSSARY

- a 九章算術  
 c 中國數學史  
 e 中國古代數學簡史  
 g 九章算術與劉徽  
 i 隋書  
 k 魏  
 m 重差  
 o 張衡  
 q 尺  
 s 實  
 u 海島算經  
 w 李淳風  
 y 欽定四庫全書  
 aa 戴震  
 ac 微波榭  
 ae 屈曾發  
 ag 海島算經細草圖說  
 ai 重差圖說  
 ak 海島算經源流考  
 am 李鐸  
 ao 天元  
 aq 楊輝算法  
 as 學藝  
 au 中國古代數學史料  
 aw 中算家的幾何學研究  
 ay 趙爽  
 ba 里  
 bc 步  
 be 表  
 bg 數書九章  
 bi 鄧文  
 bk 句股  
 bm 朱世傑  
 bo 白尚恕  
 bq 科技史文集  
 b 錢寶琮  
 d 李儼·杜石然  
 f 吳文俊  
 h 劉徽  
 j 四部叢刊  
 l 算經十書  
 n 周髀算經  
 p 靈憲  
 r 法  
 t 九章重差圖  
 v 算經十書  
 x 唐六典  
 z 永樂大典  
 ab 武英殿  
 ad 孔繼涵  
 af 李漢  
 ah 沈欽裴  
 aj 劉操南  
 al 益世報文史副刊  
 an 海島算經緯筆  
 ap 楊輝  
 ar 重差術源流及其新注  
 at 中算史論叢  
 av 許蔭舫  
 ax 海島算經古證探源  
 az 續古摘奇算法  
 bb 丈  
 bd 寸  
 bf 矩  
 bh 裴秀  
 bj 景差  
 bl 秦九韶  
 bn 四元玉鑑  
 bp 劉徽《海島算經》造術探討  
 br 我國古代測望之學重差理論  
 評介兼評數學史研究中某些  
 方法問題