Chinese computation with the counting-rods

Ang Tian Se

The counting-rods were made of bamboo. Their invention is difficult to date precisely as they could not withstand the wear and tear of time because of their perishable nature. Nevertheless, it can safely be assumed that their use during the Warring States period (480 B.C. – 221 B.C.) was extremely common.[1] Subsequently, mention of the rods was more frequent. Improvement on the construction was also made from time to time. From the Han dynasty (202 B.C. – 220 A.D.) to the Sui dynasty (581 A.D. – 618 A.D.), the length of each rod was shortened considerably to facilitate easy manipulation.[2] But towards the end of Ming dynasty (1368 – 1644), less is heard of them, as they eventually gave way to abacus.

There were two sets of counting-rod numerals current during the Ch'in and Han periods (third century B.C. onwards).[3] The sets of numerals were expressed by arrangements of bamboo sticks as follows:

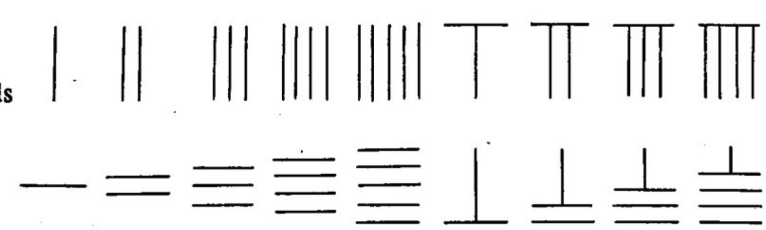
Units 1 2 3 4 5 6 7 8 9

Hundreds

Ten thousands

Tens

Thousands



(1) *For example, Lao Tzu 老子 in his Tao te ching 道德經 (in Ssu-pu ts'ung-k'an series, Shanghai, 1919), ch.27, p.10 says, "Good mathematicians do not use counting-rods".*

(2) *Ch'ien Han-shu 前漢書 (in po na pen series, Shanghai, 1931), ch. 21A, p.2a says, "......... they were bamboo sticks 1/10 ts'un in diameter and 6 ts'un long. Sui-shu 隋書 (in po na pen series, Shanghai, 1936, ch.11A, p.2b, says, "......... bamboo sticks 1/6 ts'un in diameter and 3 ts'un long."*

(3) *Needham, J., Science and civilisation in China, vol.3, Cambridge, 1959, p.8.*

An explanation for such a system is clearly stated in the Sun-tzu suan-ching 孫子算經

"In making calculations we must first know the positions of numbers. The units are vertical and the tens horizontal, the hundreds stand while the thousands lie down; thousands and tens therefore look the same, as also the ten thousands and the hundreds .... When we come to 6 we no longer pile up (rods), and the 5 has not got a one (a ligature).[4]

In operation, the digits of a number were set up on the counting-board according to their respective place-values. For example, the number 1978 appeared as 𝍩𝍨𝍯𝍧 In the case of a zero, a space was left blank.

It can be clearly seen from the above that this method of representing numerals is basically the same as the present decimal system. Furthermore, like modern written arithmetic, the numerals were arranged from left to right beginning with the highest order. This is quite a contrast to the usual way of Chinese writing which runs downwards from the top to the bottom in columns beginning from the right-hand side.

In all mathematical and astronomical works, a knowledge of the operations of addition and subtraction is taken for granted. The old masters must have considered addition and subtraction too simple to warrant any explanation. They are indeed simple, particularly when aided by counting-rods. A sum of 𝍨 and 𝍦 means taking away of the two horizontal rods of 5 making 10 while the addition of 4 and 2 vertical rods makes 6, hence the answer 16. In the case of subtraction, it literally means taking away the rods. Like the Hindu inverse process of addition mentioned by Bhaskara II in the Lilavati, the ancient Chinese method of adding was carried out from the left; the sum of the addends being placed separately below.[5] Subtraction was done in much the same way as addition except the remainder, as it was called, was placed above the larger number. In both the processes the terms 'carry' and 'borrow' had more meanings as a rod was actually lifted up and carried to the next place. If it had been 'borrowed' from the next place, it was actually 'paid back'.

In multiplication, the multiplier was placed in the upper position of the counting-board, the multiplicand in the lower position and the product in the middle position.[6] It was so arranged that the unit in the multiplicand comes just underneath the highest digit in the multiplier. The multiplicand is first

(4) *Sun-tzu suan-ching (in Chih-pu-tsu-chai ts'ung-shu series, 1777), ch. 1, p.2b.*

(5) *Illustration of this Hindu method is given in Smith D.E., History of mathematics, vol.2, Ginn, New York, 1925, p.92. See also Datta, B. and Singh, A.N., History of Hindu mathematics, Bombay, 1962, p.131.*

(6) *Sun-tzu suan-ching, ch. 1, p.2b.*

multiplied by the first digit of the multiplier, the partial product obtained is placed between the multiplicand and the multiplier. The first digit of the multiplier is then left out, and the multiplicand is drawn back by one place. When the next product is obtained from the second digit of the multiplier and the multiplicand, it is added to the previous product. Proceeding in this manner, the answer is obtained at the end of the process. The final product is actually the sum of the partial products. Thus, multiplication has been described as "an abridgment of addition".(7) The process can be followed in the accompanying diagrams set up to show the multiplication of 357 and 246. As a matter of convenience, Arabic numbers are being used instead of rod-numerals.

		2	4	6	Upper Position
					Middle Position
		3	5	7	Lower Position

(1)

(2 X 3=) 6
(2 X 5=) 10 (+
70
(2 X 7=) 14(+
714

		2	4	6
7	1	4		
3	5	7		

(2)

714
(4 X 3=) 12 (+
834
(4 X 5=) 20 (+
854
(4 X 7=) 28(+
8568

			4	6
8	5	6	8	
	3	5	7	

(3)

8568
(6 X 3=) 18 (+
8748
(6 X 5=) 30 (+
8778
(6 X 7=) 42(+
87822

				6
8	7	8	2	2
		3	5	7

(4)

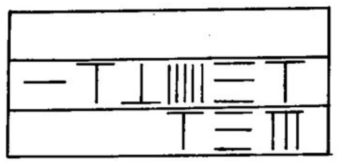
(7) *Smith, op. cit., vol.2, p.101.*

8	7	8	2	2
		3	5	7

(5)

Figure 1. The Process of Multiplication Using the Counting-rods

In division, the divisor is called the fa 法, the dividend the shih 實 and the quotient the shang 商.[8] Unlike multiplication, the quotient is arranged in the upper position. The dividend is placed in the middle position and the divisor in the lower position. Thus, 166536 ÷ 648 is arranged on the counting-board as follows:



Middle Position

Lower Position

648 is moved to the right until the first possible digit (2) in the quotient is found. Each digit in the fa (648) is then multiplied by 2. After having the product 1296 subtracted from 1665, the remainder 369 replaces 1665 in the shih position. The next step is to draw the divisor 648 to the right by one place and try to find the second digit in the quotient. Thus, the product of 648 X 5 is subtracted from 3693 which is then replaced by the difference 453. The divisor is again moved one place to the right. When the third digit of the quotient is found and proceeded in the same way, no reminder is left, showing that the division is complete.

1	6	6	5	3	6
			6	4	8

(1)

			2		
1	6	6	5	3	6
	6	4	8		

1 6 6 5
(2 X 6=) 1 2 (–
4 6 5
(2 X 4=) 8 (–
3 8 5
(2 X 8=) 1 6 (–
3 6 9

(2)

		2	5	
3	6	9	3	6
	6	4	8	

3 6 9 3
(5 X 6=) 3 0 (–
6 9 3
(5 X 4=) 2 0 (–
4 9 3
(5 X 8=) 4 0 (–
4 5 3

(3)

(8) *Sun-tzu did not give a specific name to the quotient. The quotient was called shang during the time of Hsiao-hou Yang* 夏侯陽 *(between 450 A.D. and 500 A.D.).*

	2	5	7
4	5	3	6
	6	4	8

```
             4 5 3 6
(7 X 6=)     4 2       (-
             -------
               3 3 6
(7 X 4=)       2 8     (-
             -------
                 5 6
(7 X 8=)         5 6   (-
             -------
                   0
```

(4)

	2	5	7
	6	4	8

(5)

Figure 2. The Process of Division Using the Counting-rods

Sun-tzu further commented that if the division had a remainder, the divisor (fa) must be taken as 'mother' i.e. denominator, and the remainder as 'son', i.e. numerator. However, the representation of this kind of 'mother' and 'son' relationship for a fraction was rhetorical in character. It is probable that the present method of writing common fractions is due essentially to the Hindus, although they did not use the bar. The use of the bar appears to have been an Arab development.(9)

The Chinese treatment of fractions with the counting-rods is basically the same as the modern operations. For example, the process for finding the product of $21\frac{3}{7}$ and $37\frac{5}{9}$ on the counting-board can be illustrated in the following diagrams.

The multiplicand	The integral part	2	1				(1)
	The fractional part			3		7	
The multiplier	The integral part	3	7				
	The fractional part			5		9	

1	4	7		
		3		7
3	3	3		
		5		9

(2)

1	5	0		7
3	3	8		9

(3)

(9) *Smith, op. cit., vol. 1, p.213.*

The Product (The integral part / The fractional part

The integral part		
The fractional part	5 0 7 0	6 3

(4)

8 ·0 4		
	4 8	6 3

(5)

8 0 4		
	1 6	2 1

(6)

Figure 3. Multiplication of two Mixed Numbers Using Counting-rods.

Step 3 shows the two improper fractions, while step 4 shows the product of the numerators and the product of the denominators. Step 5 is actually the reverse process of reducing the improper fractions to the mixed number. After division, the remainder is expressed in a fraction and simplified. The simplication of a fraction is not as easy as it may appear to be. The greatest common factor must first be found. In this case, both the numerator and the denominator, 48 and 63 respectively, are placed on the counting-board. The process can be followed by a series of diagrams shown below.

	(1)	(2)	(3)
The Numerator	4 8 4 8	4 8 4 8	3 3 4 8
The Denominator	6 3 6 3	1 5 6 3	1 5 6 3

(4)	(5)	(6)	(7)
3 3 4 8	1 5 4 8	1 5 4 8	1 2 4 8
1 8 6 3	1 8 6 3	3 6 3	3 6 3

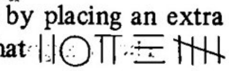
(8)	(9)	(10)	(11)
1 2 4 8	3 4 8	3 4 8	3 4 8
9 6 3	9 6 3	6 6 3	3 6 3

Figure 4. The Process of Continued Division Using the Counting-rods.

This process is referred to as the process of continued division (*keng hsiang chien sun*更相減損). Step 11 shows that alternately subtracting the smaller from the larger, the minuend is equal to the subtrahend. That is, 3 is the greatest common divisor, hence, the fraction $\frac{48}{63}$ can be reduced to $\frac{16}{21}$.

The method of computation with the counting-rods did not limit its use to the four basic arithmetic operations and the treatment of fractions only. It could also be employed in a variety of other equations, including root extraction, quadratic and cubic equations.[10]

As early as the Han period, operations with negative quantities were noted. For example, in the Fang-ch'eng 方程 (calculation by tabulation) chapter, there is a problem (Problem No. 3) on simultaneous linear equation involving both positive and negative numbers. The rule that follows this problem teaches how positive and negative numbers are being treated e.g. $a - (-b) = a + b$ etc. The positive are represented by red, and the negative by black, counting-rods. Needham avers that "this is the earliest appearance of negative quantities in any civilisation."[11] Needham further comments that the use of negative quantities was extended to quadratic equations probably by the time of Tsu Ch'ung-chih 祖沖之 (4th century A.D.) and certainly by the time of Liu I 劉益 (11th century A.D.)

The use of red and black counting-rods to denote positive and negative quantities continued in practice until the 13th century. Towards the end of the 13th century, Li Chih 李冶 indicated negative quantities by placing an extra rod obliquely over the final digit of the number concerned, so that  stands for −20734.[12]

That the use of counting-rods was a matter of daily importance in early China cannot be denied. Their use was looked upon as fundamental in the solving of problems. The size and nature of the counting-rods not only changed from dynasty to dynasty, the name itself, too, varied from time to time. The earliest term for counting-rods seemed to be ts'e 策. Later, it came to be generally known as suan-tzu 算子. Throughout the centuries, it also came to be known variously as suan 算, ch'ou 籌, ch'ou-suan 籌算, ch'ou-ts'e 籌策 and suan-ch'ou 算籌.[13]

During the Han period, a total of 271 rods constituted a set and from that it formed a hexagon that had nine rods on a side.[14] This means that they were arranged in six groups of which the ends formed a triangular number

(10) *For a review of the solutions of equations with the counting-rods, see Ang Tian Se, A study of the mathematical manual of Chang Ch'iu-chien (an unpublished M.A. thesis), University of Malaya, Kuala Lumpur, 1969, pp.124ff.*

(11) *Needham, op. cit., vol. 3, p.26. One should, however, bear in mind that the Chinese text uses words, not symbols.*

(12) *For an excellent study of Li Chih's work, see Ho Peng-yoke, "Li Chih", in Charles Coulston Gillispie ed., Dictionary of scientific biography, vol. 8, New York, 1973, pp. 313 ff. Note that the circular symbol for zero is first found in print in the Su-shu chiu-chang 數書九章 of Ch'in Chiu-shao 秦九韶 (1247 A.D.).*

(13) *For a survey of references for the term in Chinese sources, see Li Yen 李儼, Chung-kuo ku-tai shu-hs'ieh shih liao 中國古代數學史料 2nd edition, Shanghai, 1963, pp. 156–158.*

(14) *Ch'ien Han-shu, ch. 21A, p.2a.*

of 1 + 2 + ... + 9 units, or 45 in all. Six of these make 6 X 45 or 270, and these six were grouped about one central rod, making 271. This is an early example of a figurate number.(15)

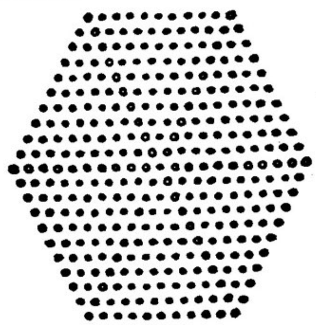


Figure 5. A Standard Set of Counting-rods

The hexagonal bundle of counting-rods as illustrated in Fig. 5 could be conveniently held in hand. More often than not, they were in a bag and carried about by the mathematicians at their girdle. The manipulation of the rods was an art and to be proficient, one required a period of training. The aim was accuracy and speed in computations. Sometimes, the promotion of minor functionaries was based primarily on one's flexibility at computations.(16) If one was well-trained in computation, one could manipulate the counting-rod with great ease. It was said that the 11th century astronomer Wei P'o 衛朴 "could move his counting-rods as if they were flying, so quickly that the eye could not follow the movements before the result was obtained."(17)

Throughout the centuries until their eventual replacement by the abacus in the 17th century, the counting-rods served as an indispensable mechanical aid to calculation. They were readily available and accessible to anyone. Naturally, the material with which the counting-rods were made was an indication of one's status. They were, however, generally made of bamboo, though they could also be of wood, paper, bone, horn, iron, ivory or jade. For example, in the third century A.D., Wang Jung 王戎, a minister of State and the patron of water-mill engineers, was said to have spent his time, day and night, reckoning his

(15) *Cheng Chin-te, "The use of computing rods in China,", The American Mathematical Monthly, vol. 32, No.10, 1925, p.494.*

(16) *See for example, a story in Kao Yen-hsiu's 高彥休 T'ang ch'ueh-shih 唐闕史 written in about the 10th century A.D., Needham, op. cit., vol. 3, p.116.*

(17) *See Shen Kua's 沈括 Meng-hsi pi-t'an 夢溪筆談, in Kuo-hsueh chi-pen ts'ung-shu series, Taipei, 1968, ch. 18, p. 118. Needham, op. cit., vol. 3, p. 72 also has the quotation, but chapter quoted is an error.*

income with his ivory counting-rods,[18] thus creating the proverbial expression "to reckon with ivory rods" 牙籌計 as an allusion to wealth.

During the early period of the Ch'ing dynasty (1644 – 1911), there was yet another type of counting-rods in widespread use by the mathematicians. These were rods graduated with numbers. According to the Ch'ou-suan 籌算 written by Mei Wen-t'ing 梅文鼎 in 1678, the graduated counting-rods were an importation from the West by the Jesuits.[19] Li Yen was even more specific and said that they had been introduced into China by Jacques Rho, known as Chinese as Lo Ya-ku 羅雅各 , in 1628.[20] To this, Needham comments that the graduated counting-rods "seem to be practically identical with Napier's bones".[21] Apparently, the graduated rods were very handy and facilitated computations considerably. They soon became vogue and seemed to catch the fancy of the literary circle as well.[22]

Tai Chen 戴震 in his Ts'e-suan 策算 (1744) gives a drawing and an explanation of the working with rods.[23] From Tai Chen's preface, it is known that the rods were made from bamboo or wood, each in the shape of a ruler. Each side of the rod was divided into nine spaces which were divided again diagonally. Each space was marked with a number; the units occupying the bottom right-hand corners while the tens in the upper left-hand corners. The first rod was numbered from 1 to 9, the second from 2 to 18, being twice the units of each number of the first rod. The numbers on the third rod are three times those of the first rod. Marking in this way, the numbers on the ninth rod are nine times those of the first rod. The tenth rod was left unnumbered. Sometimes, for portable convenience, the first and the ninth could be graduated on either side of the same rod. Similarly, the second and eighth, third and seventh, fourth and sixth could be done the same way. In this way, a set constituted a total of five rods. It was estimated that for adequate elementary operations, one required about ten sets. In addition, Tai Chen also gave a square counting-rod marked in the nine spaces with numbers from 1 to 81. Originally, all the rods were numbered with Chinese numerals from right to left. Now,

(18) *Li Fang 李昉 ed., T'ai-p'ing yu-lan 太平御覽, in Kuo-hsueh chi-pen ts'ung-shu series, Taipei, n.d., ch. 750, p. 2a.*

(19) *The text Ch'ou-suan is found in Mei-shih tsung-shu chi-yao 梅氏叢書輯要.*

(20) *Li Yen, Chung-kuo suan-hsueh hsiao-shih 中國算學小史 , Shanghai, 1939, pp. 97 – 98.*

(21) *Needham, op. cit., vol. 3, p. 72.*

(22) *A good example is seen in the early 19th century novel Ching-hua-yuan 鏡花緣 by Li Ju-chen 李汝珍. See Yu Wang-luen, "Knowledge of mathematics and science in Ching-hua-yuan", Orien Extremus, 12 (no. 2), 1974, pp. 227 – 229.*

(23) *Tai Chen's Ts'e-suan was included at the end of the Suan-ching shih-shu 算經十書, as an appendix, Kuo-hsueh ch'i-pen series, vol. 130, pp. 103 ff.*

for clarity, they are re-marked with Arabic numbers from left to right as shown below:

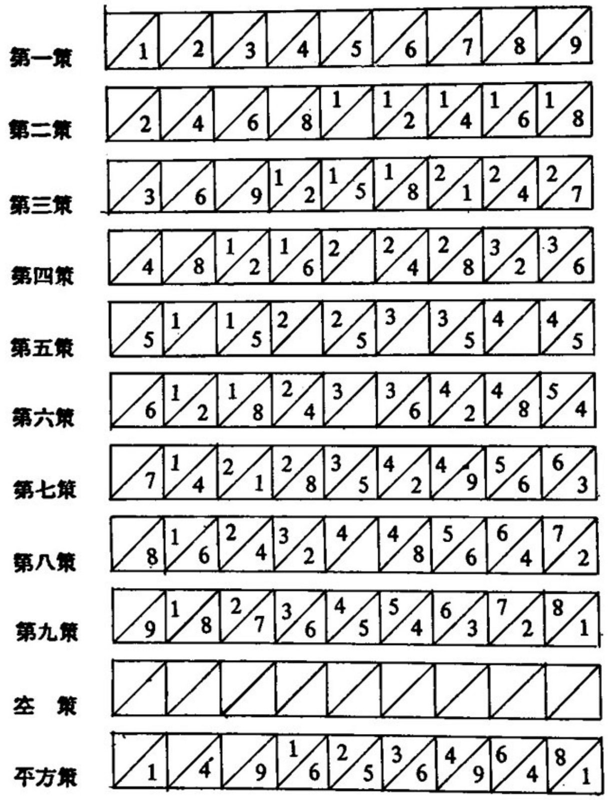
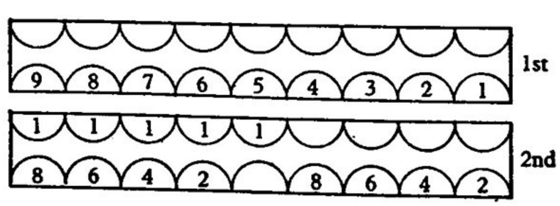


Figure 6. Tai Chen's Graduated Counting-rods

In Mei Wei-ting's Ch'ou-suan as well as Ho Meng-Yao's 何夢瑤 Suan-ti 算迪(1730), the divisions of the nine spaces on the counting-rods were in semi-circles, one above the other. For the convenience of comparison, the rods are again remarked with Arabic numbers as shown below:



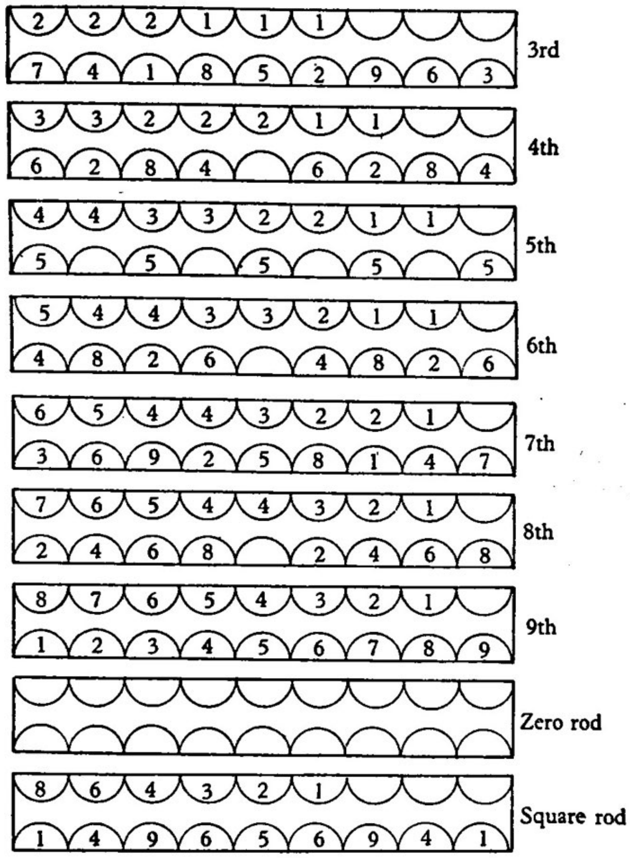
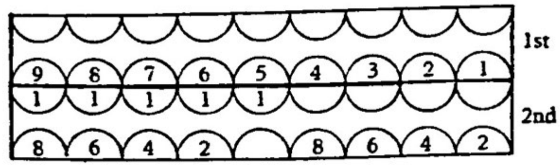


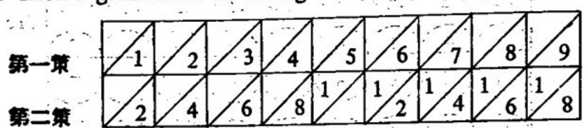
Figure 7. Mei Wen-ting and Ho Meng-yao's Graduated Counting-rods

Like the graduated rods with squared spaces, the tens occupy the upper semi-circles while the units are in the lower semi-circles. When two rods are used, the lower semi-circle of the upper rod and the upper semi-circle of the lower rod are taken to form one whole circle, and the numbers within each circle are added together so as to form the digit of that place. When two rods are used, three digits are thus formed; when three rods are used, four digits are formed and so on. For example, to multiply 357 by 12, the process of arriving at the 357 product is as follows:

As the multiplier is 12, the first and second rods are used.



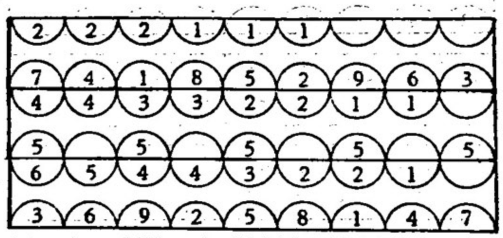
Using Tai Chen's graduated counting-rods, the arrangement is the same:



Since the multiplicand is 357, read the columns 3, 5 and 7. Adding the results together one obtains the answer as follows:

```
   36
   60
+  84
 4284
```

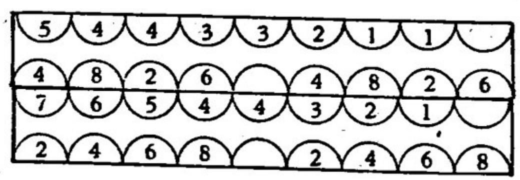
One can also consider the multiplicand first and take the third, fifth and seventh rods for operations. In this case, the rods are arranged in the following manner:



Look up the digits in the first and second spaces (for the multiplier 12) and add the digits together for the desired answer as shown below:

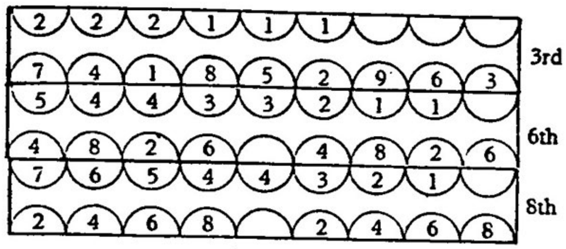
```
  357
+ 714
 4284
```

The process of division by the graduated rods is just the opposite of that of multiplication, the quotient being read from the number of the space in which the figure most nearly approximates the figure to be divided. For example, to divide 340 by 68, the sixth and eighth rods are used.



Read to the fifth space, the number is 340; subtracting, 340 – 340 = 0; and so 5 is the quotient.

If a number is not exactly divisible by another, the rule mentioned earlier on still holds good. For example, the process of finding the quotient of the division of 4574 by 368 to three places of decimal can be illustrated as follows:



Inspect all the spaces vertically for a figure close to 4574. The figure in the first space is a satisfactory approximation. Carrying out the continuous subtraction yields the following result.

4574	....	dividend
– 368	....	figure in the lst space
894		
– 736	....	figure in the second space
158		

It is clear from the above that 12 is the integral quotient with a remainder 158. To reduce the remainder to three places of decimal, continue the inspection as before:

158	....	remainder
– 147.2	....	figure in the 4th space
10.8		
– 7.36	....	figure in the 2nd space
3.44		
– 3.312	....	figure in the 9th space
0.128		

The process seems laborious. Yet with a sense of place-value and bearing in mind the uncanny dexterity of manipulation, the job could be easily accomplished by any skilful Chinese mathematician.(24) Nevertheless, this art of computation died a natural death when it was finally replaced by the abacus.

(24) *For further reading on this art of computation, see Hsu Ch'un-fang* 許純舫, *Chung-kuo suan-shu ku-shih* 中國算術故事 *Peking, 1965, pp. 51 ff; Cheng Chin-te, loc. cit., pp. 496 ff.*